UNSTEADY PROPAGATION OF LONGITUDINAL SHEAR CRACKS

(NEUSTANOVIVSHEESIA RASPROSTRANENIE TRESHCHINY PRODOL'NOGO SDVIGA)

PMM Vol.30, № 6, 1966, pp. 1042-1049

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(Received April 11, 1966)

In all problems of unsteady crack propagation which have been solved to date [1 to 3], it has been assumed that the crack propagates at a constant speed. This assumption was not prompted by physical considerations of the problem, but by the methods of solution, therefore, the applicability of the results is limited. It would be more realistic to consider the speed of crack propagation as a function of time based on explicit physical hypotheses. Unfortunately, the general case of the resultant problem cannot be solved by existing methods. However, the problem of longitudinal shear cracks, i.e. the plane problem in which the displacement is parallel to the crack boundary, may be solved for an arbitrary given variation in crack propagation speed, utilizing the method developed in connection with the theory of supersonic flows [4 and 5].

Note that equilibrium problems of longitudinal shear cracks have been studied in [6 and 7].

1. Formulation of problem. Consider an infinite elastic body whose shear modulus $\mu = 1$, speed of transverse wave propagation $b = 1$ and which occupies the space outside of the crack (Fig.1), given by

$$x_1 < x < x_2, \quad -\infty < y < \infty, \quad z = 0 \tag{1.1}$$

Assume that all loads applied to the body are directed along y and are constant along this axis. Then the displacement vector will also be in the y direction. Let w denote the single component of this vector. The stress tensor has only two non-zero components

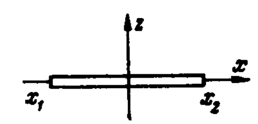

Fig. 1

$$\tau_{xy} = \partial w / \partial x, \qquad \tau_{yz} = \partial w / \partial z \tag{1.2}$$

All of the above quantities are not functions of y. We assume further that the state of stress of the body is such that w is an odd function of z. By the principle of superposition, we may separate the terms due to an initial state of stress and the terms due to body forces in the absence of a crack, thus reducing any arbitrary problem with zero initial conditions and specified loads along the crack

$$w \equiv 0, \quad \partial w / \partial t \equiv 0, \quad \text{for} \quad t = 0 \tag{1.3}$$

$$\tau_{yz} = -p(x, t) \quad \text{for } x_1 < x < x_2, \ z = 0 \tag{1.4}$$

For the time being, we will study the problem under the assumption that the crack boundary is specified at some instant

$$x_1 = x_1(t), \qquad x_2 = x_2(t) \tag{1.5}$$

where $x_2(t)$ and $x_1(t)$ are, respectively, monotonously increasing and decreasing functions of time, i.e.

$$x_1^{\cdot}(t) < 0, \qquad x_2^{\cdot}(t) > 0, \qquad t > 0 \tag{1.6}$$

the dot denoting differentiation with respect to time.

The enumerated conditions must be accompanied by the equation of motion

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z}$$

which is equivalent to the wave equation in w

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \tag{1.7}$$

In view of the assumed symmetry of the problem, it is sufficient to obtain a solution in the half-plane $z > 0$ only; at $z = 0$ this solution is to satisfy, in addition to (1.4), the conditions

$$w = 0 \quad \text{for } z = 0, \qquad -\infty < x < x_1, \quad x_2 < x < \infty \tag{1.8}$$

since w is continuous outside the crack, and is an odd function of z.

2. Solution for τ_{yz} for $z = 0$. Hereinafter the subscripts for τ_{yz} will be omitted, i.e. we will write τ instead of τ_{yz}.

Using the method of Volterra, we readily obtain the relation

$$w(x_0, z_0, t_0) = \frac{1}{\pi} \iint\limits_S \frac{\tau(x, t)\, dx\, dt}{\sqrt{(t_0 - t)^2 - (x_0 - x)^2 - z_0^2}} \qquad (\tau(x, t) \equiv \tau(x, 0, t)) \tag{2.1}$$

valid for $z_0 \geq 0$. Here S is that part of the xt plane which lies inside the cone

$$(t_0 - t)^2 - (x_0 - x)^2 - z_0^2 \geqslant 0, \qquad 0 \leqslant t \leqslant t_0 \tag{2.2}$$

For $z_0 = 0$ we obtain

$$w(x_0, t_0) = \frac{1}{\pi} \iint\limits_{S_0} \tau(x, t) \frac{dx\, dt}{\sqrt{(t_0 - t)^2 - (x_0 - x)^2}} \tag{2.3}$$

Here S_0 is the triangle

$$(t_0 - t)^2 - (x_0 - x)^2 \geqslant 0, \qquad 0 \leqslant t \leqslant t_0 \tag{2.4}$$

By virtue of (1.8) we have

$$\iint\limits_{S_0} \tau(x, t) \frac{dx\, dt}{\sqrt{(t_0 - t)^2 - (x_0 - x)^2}} = 0 \tag{2.5}$$

for $x_0 < x_1(t)$ or $x_0 > x_2(t)$.

Let us first examine the time interval $0 < t_0 < x_0 - x_1(0)$, when the disturbances from the left edge of the crack have not yet reached the observation point, for $x_0 > x_2(t)$. Clearly, from the homogeneity of the initial conditions, $\tau(x, t) \equiv 0$ for $x > t + x_2(0)$. Consequently, the region of integration for this case is defined by

$$t_0 - t \geqslant |x_0 - x|, \qquad t > x - x_2(0)$$

The subregion defined by $x < x_2(t)$ will be denoted by S_1 ; the subregion defined by $x > x_2(t)$ will be denoted by S_2 . On S_1, $\tau(x, t)$ is given by (1.4). Thus

$$\iint_{S_2} \tau(x, t) \frac{dxdt}{\sqrt{(t_0 - t)^2 - (x_0 - x)^2}} = \iint_{S_1} p(x, t) \frac{dxdt}{\sqrt{(t_0 - t)^2 - (x_0 - x)^2}} \tag{2.6}$$

Here the right-hand side is a known function of x_0 and t_0 .

Let us introduce the characteristic coordinates of the system

$$\xi = (t - x) / \sqrt{2}, \qquad \eta = (t + x) / \sqrt{2} \tag{2.7}$$

Then (2.6) takes the form

$$\int_{-x_2(0)/\sqrt{2}}^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}} \int_{\eta_2(\xi)}^{\eta_0} \tau_1(\xi, \eta) \frac{d\eta}{\sqrt{\eta_0 - \eta}} = \int_{-x_2(0)/\sqrt{2}}^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}} \int_{-\xi}^{\eta_2(\xi)} p_1(\xi, \eta) \frac{d\eta}{\sqrt{\eta_0 - \eta}}$$

$$(\tau_1(\xi, \eta) \equiv \tau(x, t), \quad p_1(\xi, \eta) \equiv p(x, t)) \tag{2.8}$$

The function $\eta_2(\xi)$ is a solution of Equation

$$\eta_2 - \xi = \sqrt{2} x_2 \left(\frac{\eta_2 + \xi}{\sqrt{2}} \right) \tag{2.9}$$

i.e. Equation $\eta = \eta_2(\xi)$ defines the position of the right rdge of the crack in terms of ξ and η .

Clearly, (2.8) will be satisfied if

$$\int_{\eta_2(\xi)}^{\eta_0} \tau_1(\xi, \eta) \frac{d\eta}{\sqrt{\eta_0 - \eta}} = \int_{-\xi}^{\eta_2(\xi)} p_1(\xi, \eta) \frac{d\eta}{\sqrt{\eta_0 - \eta}} \tag{2.10}$$

The above is in the form of Abel's integral equation in $\tau_1(\xi, \eta)$. Its solution is given by

$$\tau_1(\xi_0, \eta_0) = \frac{1}{\pi \sqrt{\eta_0 - \eta_2(\xi_0)}} \int_{-\xi_0}^{\eta_2(\xi_0)} p_1(\xi_0, \eta) \frac{\sqrt{\eta_2(\xi_0) - \eta}}{\eta_0 - \eta} d\eta \tag{2.11}$$

The preceding expression holds for $\eta_0 > \eta_2(\xi_0)$, i.e. to the right of the crack. The stress to the left of the crack may be obtained in a similar manner, and is given by

$$\tau_1(\xi_0, \eta_0) = \frac{1}{\pi \sqrt{\xi_0 - \xi_1(\eta_0)}} \int_{-\eta_0}^{\xi_1(\eta_0)} p_1(\xi, \eta_0) \frac{\sqrt{\xi_1(\eta_0) - \xi}}{\xi_0 - \xi} d\xi \tag{2.12}$$

for $\xi_0 > \xi_1(\eta_0)$, i.e. to the left of the crack. Here $\xi_1(\eta)$ is the solution of Equation

$$\eta - \xi_1 = \sqrt{2} x_1 \left(\frac{\eta + \xi_1}{\sqrt{2}} \right) \tag{2.13}$$

In terms of physical variables, (2.11) and (2.12) take the form

$$\tau(x_0, t_0) = \frac{1}{\pi \sqrt{x_0 - x_2(t_2)}} \int\limits_{x_0 - t_0}^{x_2(t_2)} p(x, t_0 - x_0 + x) \frac{\sqrt{x_2(t_2) - x}}{x_0 - x} dx \quad (2.14)$$

for $x_0 > x_2(t_0)$, where t_2 is the solution of

$$t_0 - x_0 = t_2 - x_2(t_2) \quad (2.15)$$

for $x_0 < v x_1(t_0)$

$$\tau(x_0, t_0) = \frac{1}{\pi \sqrt{x_1(t_1) - x_0}} \int\limits_{x_0 + t_0}^{x_1(t_1)} p(x, t_0 + x_0 - x) \frac{\sqrt{x - x_1(t_1)}}{x_0 - x} dx \quad (2.16)$$

where t_1 is the solution of

$$t_0 + x_0 = t_1 + x_1(t_1) \quad (2.17)$$

Expressions (2.14) and (2.16) have been obtained for the time intervals $0 < t_0 < x_0 - x_1(0)$ and $0 < t_0 < x_2(0) - x_0$, respectively. Moreover, the intervals of integration for both expressions lie entirely on the crack surface. To determine $\tau(x_0, t_0)$ for larger values of time it is necessary to interchange (2.14) and (2.16), setting $p(x, t) \equiv -\tau(x, t)$ on those portions of range of integration on which the stress is unknown. This procedure corresponds to repeated diffraction of the waves at the crack boundary.

3. Coefficient of stress intensity. Expressions (2.14) and (2.16) give infinite values for the stresses at the crack boundary. In the neighborhood of the right edge of the crack, (2.14) yields the following asymptotic expression for $\tau(x_0, t_0)$:

$$\tau(x_0, t_0) \approx \frac{k_2}{\pi \sqrt{x_0 - x_2(t_0)}} \qquad \text{for } x_0 \to x_2(t_0) \quad (3.1)$$

Here k_2 is the coefficient of stress intensity at the right edge of the crack

$$k_2 = \sqrt{1 - x_2^{\cdot}(t_0)} \int\limits_{x_2(t_0) - t_0}^{x_2(t_0)} p(x, t_0 - x_2(t_0) + x) \frac{dx}{\sqrt{x_2(t_0) - x}} \quad (3.2)$$

Similarly

$$\tau(x_0, t_0) \approx \frac{k_1}{\pi \sqrt{x_1(t_0) - x_0}} \qquad \text{for } x_0 \to x_1(t_0) \quad (3.3)$$

where

$$k_1 = \sqrt{1 + x_1^{\cdot}(t_0)} \int\limits_{x_1(t_0)}^{x_1(t_0) + t_0} p(x, t_0 + x_1(t_0) - x) \frac{dx}{\sqrt{x - x_1(t_0)}} \quad (3.4)$$

If the transformation is made to a moving coordinate system with the origin at the edge of the crack and the x-axis along the crack, then the following expression is obtained, which is valid for both the right and left edge of the crack:

$$k = \sqrt{1 - v} \int\limits_0^t p(x', t - x') \frac{dx'}{\sqrt{x'}} \quad (3.5)$$

where $x' > 0$ on the crack, while $x' < 0$ on its extension, and v is the velocity of displacement of the crack edge. For the left edge of the crack $x' = x - x_1(t)$, $v = -x_1^{\cdot}$; for the right edge $x' = x_2(t) - x$, $v = x_2^{\cdot}$.

With the aid of (3.5) we may carry over some of the results of static crack theory to the dynamic case. Let $p_0(x, t)$ be the distribution of the cohesive forces in the neighborhood of the crack boundary. Then the coefficient of stress intensity which takes into account the cohesive forces is given by

$$k' = \sqrt{1-v}\int_0^t [p(x', t-x') - p_0(x', t-x')]\frac{dx'}{\sqrt{x'}}$$

We now require that the stresses at the crack boundary be finite (Khristianovich-Barenblatt condition [8]). Thus, $k' = 0$. Defining the modulus of cohesion

$$K(v, t) = \sqrt{1-v}\int_0^t p_0(x', t-x')\frac{dx'}{\sqrt{x'}} \tag{3.6}$$

we obtain the condition

$$k = K(v, t) \tag{3.7}$$

where k is the coefficient of stress intensity obtained without regard to the coehesive forces.

Let the length of the crack boundary zone equal l . Generally, we can consider l to be very small in comparison with any characteristic dimension of the problem. Consider the case $t \gg l$. Then (3.6) may be simplified by neglecting x' in comparison with t , yielding

$$K(v, t) = \int_0^l p_0(x', t)\frac{dx'}{\sqrt{x'}}\sqrt{1-v} \tag{3.8}$$

Now assume that the distribution of cohesive forces at the boundary of the propagating crack depends only on the speed of crack propagation, and is not an explicit function of time. This assumption is a natural generalization of Barenblatt's hypothesis on the independent character of the crack boundary zone. In that case, the modulus of cohesion will only be a function of v , and be given by

$$K(v) = \int_0^l p_0(x', v)\frac{dx'}{\sqrt{x'}}\sqrt{1-v} \tag{3.9}$$

4. Energetic condition. The cohesive forces infuence the stress distribution in the body only at a distance of order l from the crack boundary. Thus, for small l , the boundary zone may be taken as a point, and the cohesive forces may be disregarded. Then, of course, the condition that the stresses be finite cannot be fulfilled, but another condition may be obtained, defining the coefficient of stress intensity. Namely, we can assume that the work done in the rupture process (in overcoming the cohesive forces) depends only on the speed of crack propagation, i.e. for a given material, it may be expressed as a function of the speed of propagation

$$P = P(v) \tag{4.1}$$

Consider the energy integral obtained from the equation of motion; this can be related to the coefficient of stress intensity by

$$P(v) = \pi^{-1}(1-v^2)^{-1/2}k^2 \tag{4.2}$$

where k is given by (3.2) or (3.4). Relation (4.2) may be rewritten as

$$k \equiv K(v) = \sqrt{\pi P(v)\sqrt{1-v^2}} \tag{4.3}$$

As $v \to 0$, this condition becomes Griffith's condition for a static longitudinal shear crack.

The function (4.1) has to be determined either experimentally or theoretically from some physical assumptions regarding the rupture mechanism. For example, if we assume that in the progress of crack formation no plastic deformation takes place, and all work is spent in increasing the surface energy, then

$$K(v) = \sqrt{2\pi T \sqrt{1-v^2}} \tag{4.4}$$

where T is the surface tension, which is a material constant. Experimentally, it is convenient to determine the coefficient of stress intensity, or in the final analysis, the cohesion modulus $K(v)$, rather than the function in (4.1).

Now it is no longer necessary to consider $x_1(t)$ and $x_2(t)$ as known functions of time. Substituting (3.2) and (3.4) into (3.7), we obtain a differential equation for the determination of the locations of the crack ends at any time; for example

$$\int_{x_2-t}^{x_2} p(x, t - x_2 + x) \frac{dx}{\sqrt{x_2 - x}} = \frac{K(x_2^{\cdot})}{\sqrt{1 - x_2^{\cdot}}} \tag{4.5}$$

5. Examples. The investigation of problems for cracks of finite length can only be conducted numerically, since the multiple integrals in connection with the repreated wave diffractions cannot be obtained in closed form even in the simplest cases, Therefore, the examples considered below are for a semi-infinite crack only. Actually, the results obtained in the case of a semi-infinite crack are also applicable to finite cracks at such times that the disturbances from one end of the crack have not yet reached the other end.

a) Consider the case of an elastic continuum which is initially under a homogeneous state of stress such that $\tau_{yz} = \tau_0$; at $t = 0$, an instantaneous semi-infinite crack develops along the negative x-axis. In that case $p = \tau_0$ and is independent of x and t .

Substitution of the above value into (3.2) yields the coefficient of stress intensity

$$k = 2\sqrt{1 - x^{\cdot}}\, \tau_0 \sqrt{t} \tag{5.1}$$

where $x(t)$ is the coordinate of the crack end at time t . So long as the magnitude of the stress intensity coefficient has not reached the value of the static cohesion modulus, the crack remains stationary, i.e.

$$x = 0 \qquad \text{for } t < t^\circ = [K(0)]^2 / 4\tau_0^2 \tag{5.2}$$

Crack propagation starts at time $t = t^\circ$, and k must then be equal to the cohesion modulus $K(x^{\cdot})$, i.e.

$$K(x^{\cdot})(1 - x^{\cdot})^{-1/2} = 2\tau_0 \sqrt{t} \tag{5.3}$$

This differential equation determines $x(t)$. If $K(x^{\cdot})$ is bounded, then

$$x^{\cdot} \to 1 \qquad \text{for } t \to \infty$$

i.e. the velocity of crack propagation approaches, with time, the transverse wave velocity of the medium, and the crack propagation never stops. This is only natural, since there exists no solution for a semi-infinite equilibrium crack in a homogeneous stress field.

b) As a second example, consider a semi-infinite crack with a concentrated load $p = p^\circ \delta(x + x^\circ)$ applied at time $t = 0$ at the point $x = -x^\circ$. In this case, (3.2) yields

$$k = \frac{\sqrt{1 - x^{\cdot}}}{\sqrt{x + x^\circ}} H(t - x^\circ)\, p^\circ \tag{5.4}$$

where $H(t)$ is the Heaviside function. This expression is zero for $t < x^\circ$ i.e. so long as the disturbance has not yet reached the crack end. Thus, the crack does not propagate, and

$$x = 0 \quad \text{for} \quad t < x^\circ \tag{5.5}$$

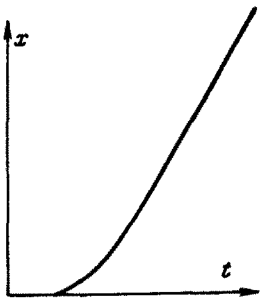

Fig. 2

At the time $t = x^\circ$, the magnitude

$$k = \frac{p^\circ}{\sqrt{x^\circ}} \sqrt{1 - x^{\cdot}(0)} \leqslant \frac{p^\circ}{\sqrt{x^\circ}} \tag{5.6}$$

If this quantity is less than the static cohesion modulus, the crack will not begin to propagate at all, since it is clear from (5.4) that k can only decrease in the course of crack propagation. Hence, crack propagation will take place only under conditions

$$\frac{p^\circ}{\sqrt{x^\circ}} > K(0)$$

In that case, equating the expression in (5.4) to the dynamic cohesion modulus, we obtain a differential equation for $x(t)$

$$K(x^{\cdot}) \frac{\sqrt{x + x^\circ}}{\sqrt{1 - x^{\cdot}}} = p^\circ \tag{5.7}$$

Again, assuming $K(x^{\cdot})$ to be bounded we conclude that the crack will only propagate as long as its boundary has not reached the point

$$x_m = [p^\circ / K(0)]^2 - x^\circ \tag{5.8}$$

after which crack propagation will cease.

In both of the investigated problems, more detailed information may be obtained with regard to crack propagation if some definite form is assumed for $K(v)$, and Equations (5.3) and (5.7) are integrated. For definiteness, assume that the energy of rupture is constant, i.e. we will consider a purely brittle fracture with no plastic deformation. In this case, the cohesion modulus is given by (4.4). Substituting this expression into (5.3) and (5.7) we obtain the governing equation of motion for the crack

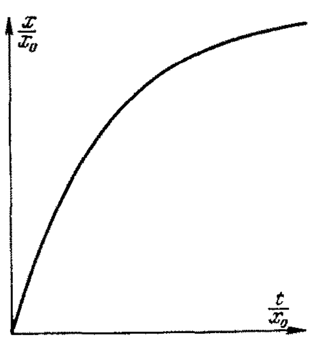

Fig. 3

for example (a)

$$x = t + \left(\frac{\pi}{2} - 1 - 2 \tan^{-1} \frac{t}{t^\circ}\right) t^\circ$$

and for example (b)

$$t = x + (x_m + x^\circ) \ln \left| \frac{x_m (x_m + 2x^\circ - x)}{(x_m + 2x^\circ)(x_m - x)} \right|$$

The character of the corresponding curves is shown in Figs.2 and 3.

The formulas obtained herein, in principle enable one to study the propagation of longitudinal shear cracks under arbitrary loading and over arbitrary time intervals, provided that the functional relationship between the cohesion modulus and the speed of crack propagation is known. The only limitation is that the initial crack length must be finite and large in comparison with the boundary region in order that the idea of cohesion modulus be meaningful. Investigation of the initial period of the crack propagation before interaction between the boundaries occurs, is particularly simple. Afterwards, analysis becomes increasingly complex as time increases, and can be carried out only by numerical methods.

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Translated by H.H.